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Evolution of a gauge-invariant measure of the isocharge in the far field of a 'tHooft–Polyakov monopole

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Abstract. We consider the dynamics of a point-like SU(2) isocharge which creates a weak disturbance in the field of a 'tHooft-Polyakov monopole. We introduce a local, gauge-invariant measure of the isocharge, and we compute its evolution, for various motions, to leading order in a small paramter ε, where ε measures the weakness of the disturbance. In contrast to ordinary electromagnetic charge, the gauge-invariant isocharge need not remain constant, but rather undergoes a net change, for certain motions, as a result of its interaction with the monopole.

1. Introduction and statement of the problem

Despite the importance of non-Abelian monopoles in elementary particle physics, relatively little is known about the two-body problem. Motion and radiation reaction in non-Abelian gauge theories have been studied by various authors, including Drechsler, Havas, Kates and Rosenblum [1-4] (and papers cited therein), but progress on the dynamics of classical non-Abelian monopoles has been slow, due to the nonlinear nature of the problem.

A significant issue in classical Yang-Mills theories has been the construction of a gauge-invariant measure for the isocharge associated with a particular Yang-Mills field configuration. The definitions of isocharge given in the literature [5-9] depend on asymptotic properties of the fields and do not lend themselves to a local interpretation. In the present problem, we will be interested in the time evolution of a point-like isocharge moving along various worldlines $\Gamma(\tau)$ and interacting with a monopole-like object. The introduction of a Higgs field [10] Φ^a provides a preferred direction in the local SU(2) fibre or 'isospace', as we will call it here. If the current of our moving isocharge is assumed to take the point-like form

$$j^{a\nu} = \int \mathrm{d}\tau\, q^a(\tau) U^\nu \delta[x^\mu - R^\mu(\tau)] \tag{1.1}$$

then the quantity $q \equiv q^a\Phi^a|_\Gamma$ is independent of gauge. (Spacetime indices are designated by Greek letters; group and spatial indices by Latin ones.)

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The classical SU(2) Yang-Mills-Higgs theory [11, 12]

$$\mathcal{L} = -\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \tfrac{1}{2}(D_\mu\Phi^a)(D^\mu\Phi^a) - \tfrac{1}{4}\lambda(\Phi^a\Phi^a - F^2)^2 - 4\pi j^a_\mu A^{a\mu} \quad (1.2)$$

$$G^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a{}_\mu + g\varepsilon^{abc}A_\mu{}^b A_\nu{}^c \qquad D_\mu\Phi^a = \partial_\mu\Phi^a + g\varepsilon^{abc}A^b{}_\mu\Phi^c \quad (1.3)$$

with field equations

$$D_\mu G^{a\mu\nu} - g\varepsilon^{abc}(D^\nu\Phi)^b = 4\pi j^{a\nu} \quad (1.4)$$

$$D_\mu D^\mu\Phi^a + \lambda(\Phi^b\Phi^b)\Phi^a - \lambda F^2\Phi^a = 0 \quad (1.5)$$

and covariant charge current identities

$$D_\mu j^{a\mu} = 0 \quad (1.6)$$

$$D_\mu J^{a\mu} = 0 \qquad 4\pi J^{a\mu} \equiv g\varepsilon_{abc}\Phi^b(D^\nu\Phi)^c \quad (1.7)$$

has a well known class of solutions [13, 14]

$$\Phi^a = \frac{x^a}{r}F(r) \quad (1.8)$$

$$A^a{}_i = \varepsilon_{aij}\frac{x^j}{r}W(r) \quad (1.9)$$

$$A^a{}_0 = 0 \quad (1.10)$$

where $W(r)$ and $F(r)$ have the asymptotic forms [15, 12]

$$W(r) \to \frac{1}{gr} - \frac{A_0}{g}\exp(-gFr) \quad (1.11)$$

$$F(r) \to \begin{cases} F - D_0[\exp(-\sqrt{2}\,\lambda Fr)]/(gr) & \lambda > 0 \\ F - 1/(gr) & \lambda = 0. \end{cases} \quad (1.12)$$

The monopole is of course a solution of the *homogeneous* system. This field represents a magnetic monopole of (magnetic) charge $1/g$ centred at the origin. In particular, in the 'limit' $\lambda \to 0$, the functions $W(r)$ and $F(r)$ can be found analytically [16]:

$$W(r) = \frac{1}{gr} - \frac{F}{\sinh(gFr)} \quad (1.13)$$

$$F(r) = \frac{F}{\tanh(gFr)} - \frac{1}{gr}. \quad (1.14)$$

For any λ, the monopole exhibits a characteristic reference length $1/(gF)$. Without loss of generality, let us choose a system of coordinates in which $g = 1$, $F = 1$, which means that distances are measured in units of $1/(gF)$ and charges in units of $1/g$.

Consider now a disturbance generated by the addition of a small point-like isocharge. By 'small' isocharge, we mean that the charge-current vector can be written in the form

$$j^{a\mu} = \varepsilon\int_{-\infty}^{\infty} Q^{a\mu}(\tau)\delta^4(x^\mu - R^\mu(\tau))\,\mathrm{d}\tau \quad (1.15)$$

$$Q^{a\mu}(\tau) = Q^a(\tau)V^\mu(\tau) \qquad [V^\mu(\tau) \equiv \mathrm{d}R^\mu(\tau)/\mathrm{d}\tau] \quad (1.16)$$

where ε is a small, dimensionless parameter. Our assumption of a 'small' isocharge leads us to seek asymptotic expansions of the fields in the form

$$A^{a\mu} \sim M^{a\mu} + \varepsilon T^{a\mu} + \dots \tag{1.17}$$

$$\Phi^{a\mu} \sim f^a + \varepsilon \psi^a + \dots \tag{1.18}$$

where $M^{a\mu}$ and f^a represent the unperturbed 'tHooft-Polyakov [13, 14] solution expressed in our non-dimensionalised coordinates, i.e.

$$f^a \equiv \frac{x^a}{r} F(r) \tag{1.19}$$

$$M^{ai} \equiv \varepsilon_{aij} \frac{x^j}{r} W(r) \tag{1.20}$$

$$M^{a0} = 0. \tag{1.21}$$

The asymptotic behaviour of W and F is given by

$$W \to \frac{1}{r} - A_0 \exp(-r) \qquad (r \to \infty) \tag{1.22}$$

$$F \to 1 - (D_0/r) \exp(-\sqrt{2\lambda}\, r) \qquad (r \to \infty) \tag{1.23}$$

where A_0 and D_0 are constants. The isocharge $Q^a(\tau)$ is regarded as depending implicitly on ε. (Such expansions will of course be non-uniform near the singular worldline. Several techniques, such as Riesz potentials [4] and matched asymptotic expansions [17], could be applied to study this region more closely, but for the purposes of this paper one can do without them.) Our procedure will be to substitute these expansions in the field equations (1.4) and (1.5) and the covariant current conservation identity (1.6) and to collect terms of like power in ε. The zeroth order is of course satisfied identically by the 'tHooft-Polyakov [14, 15] solutions. The $O(\varepsilon)$ terms give *linear* equations for $T^{a\mu}$ and ψ^a as well as the $O(\varepsilon)$ covariant current conservation identity

$$\partial_\nu j^{a\nu} + M^{b\nu} j^c{}_\nu \varepsilon_{abc} = 0. \tag{1.24}$$

However, the results of this section do not in fact require the direct application of the $O(\varepsilon)$ field equations (1.4), but only of their consequence, the covariant current conservation identity (1.6). (The neglect of terms beyond $O(\varepsilon)$ in this paper means physically that we are neglecting the non-Abelian interaction of the disturbance with itself, but including its interaction with the monopole.)

2. Evolution of gauge-invariant isocharge

Let us now consider in detail the consequences of the covariant current conservation identity (1.6). Substituting the assumed form (1.1) of the current into (1.6) yields the evolution equation

$$\frac{dQ^a}{d\tau} = -\varepsilon_{abc} M^{b\nu}|_\Gamma V_\nu Q^c = \left.\frac{W(r)}{r}\right|_\Gamma (R^c V^a - R^a V^c) Q^c. \tag{2.1}$$

We are interested in the evolution of the gauge invariant isocharge scalar

$$Q(\tau) \equiv Q^a(\tau) \Phi^a(R^\nu(\tau)). \tag{2.2}$$

At the present level of approximation, $Q(\tau)$ changes along the worldline $\Gamma(\tau)$ at the rate

$$\frac{\mathrm{d}Q}{\mathrm{d}\tau}=\left(\frac{\mathrm{d}Q^a}{\mathrm{d}\tau}f^a+Q^aV^\mu\partial_\mu f^a\right)\bigg|_\Gamma. \tag{2.3}$$

Combining equations (1.19)-(1.23) and (2.1)-(2.3), one obtains

$$\frac{\mathrm{d}Q}{\mathrm{d}\tau}=\left[F(r)\left(\frac{1}{r}-W(r)\right)\right]\bigg|_\Gamma Q^cV^c+\left[\frac{F(r)W(r)}{r^2}+\frac{1}{r}\partial_r\left(\frac{F(r)}{r}\right)\right]\bigg|_\Gamma R^cR^aV^aQ^c. \tag{2.4}$$

Consider now a source whose distance of closest approach to the origin satisfies $1/b \ll 1$ (in conventional units, $1/(bFg) \ll 1$). Our first main result now follows from substitution of the asymptotic forms (1.11) and (1.12) into the isocharge evolution equation (2.4) and expansion in $1/b$. We distinguish the cases $\lambda=0$ (Prasad-Sommerfield [16] and $\lambda \neq 0$. For $\lambda \neq 0$,

$$\frac{\mathrm{d}Q}{\mathrm{d}\tau}\sim \mathrm{O}[\exp(-1/b), \exp(-\sqrt{2\lambda}\, b)/b]\mathrm{O}(\varepsilon). \tag{2.5}$$

According to equation (2.5), for *any* motion confined to the region $b \gg 1$, the scalar isocharge remains constant up to *exponentially* small corrections. For $\lambda=0$,

$$\frac{\mathrm{d}Q}{\mathrm{d}\tau}\sim \mathrm{O}\left(\frac{1}{b^2}\right)\mathrm{O}(\varepsilon) \tag{2.6}$$

and in this case the scalar isocharge is subject to changes that fall off only as $1/b^2$. Notice, however, that in both cases ($\lambda=0$, $\lambda\neq 0$) the leading terms of equation (2.4), which are of $\mathrm{O}(1/b)$, vanish.

3. Vanishing net change of isocharge in linear motion

We next consider the case of forced uniform straight-line motion for *any* impact parameter b. We orient our coordinate system such that the particle is moving along the z direction in the xz plane. The worldline $\Gamma(\tau)$ is specified by

$$R^\mu(\tau)=\gamma(\tau, b, 0, v\tau) \tag{3.1}$$

$$V^\mu(\tau)=\gamma(1,0,0,v) \tag{3.2}$$

$$\gamma\equiv(1-v^2)^{-1/2}. \tag{3.3}$$

Substituting the assumed motion (3.1) into the isocharge evolution equation (2.1), we obtain the following system:

$$\frac{\mathrm{d}Q^x}{d\tau}=-\frac{W}{R}vb\gamma^2Q^z \tag{3.4}$$

$$\frac{\mathrm{d}Q^y}{\mathrm{d}\tau}=0 \tag{3.5}$$

$$\frac{\mathrm{d}Q^z}{\mathrm{d}\tau}=\frac{W}{R}vb\gamma^2Q^x \tag{3.6}$$

with the general solution

$$Q^x = A\cos\Theta \tag{3.7}$$

$$Q^y = \text{constant} \tag{3.8}$$

$$Q^z = A\sin\Theta \tag{3.9}$$

$$\Theta \equiv vb\gamma^2 \int_{\tau_0}^{\tau} \mathrm{d}\tau \frac{W[R(\tau)]}{R(\tau)}. \tag{3.10}$$

Choosing the initial time such that $\tau_0 = 0$, we note that $Q^x(\tau)$ and $Q^z(\tau)$ are even and odd functions, respectively, of the proper time. Substituting the above into the evolution equation (2.4) for the scalar isocharge, we obtain

$$\frac{\mathrm{d}Q}{\mathrm{d}\tau} = A\{\sin\Theta[h(R)v^3\tau^2\gamma^2 + f(R)v\gamma] + \cos\Theta[h(R)bv^2\gamma^3\tau] \tag{3.11}$$

where

$$f(R) \equiv F(R)\left(\frac{1}{R} - W(R)\right) \tag{3.12}$$

$$h(R) \equiv \frac{F(R)\,W(R)}{R^2} + \frac{1}{R}\frac{\partial}{\partial R}\left(\frac{F(R)}{R}\right). \tag{3.13}$$

From equation (3.11) it is evident that $\mathrm{d}Q/\mathrm{d}\tau$ is an odd function of τ, and therefore the *net* change:

$$\Delta Q \equiv \int_{\tau=-\infty}^{\tau=\infty} \frac{\mathrm{d}Q}{\mathrm{d}\tau}\,\mathrm{d}\tau \tag{3.14}$$

vanishes.

4. Secular evolution of scalar isocharge in circular motion

Consider now the case of forced uniform circular motion. The position and four-velocity vectors are thus assumed to take the form

$$R^\mu = (t, \boldsymbol{R}) = (\gamma\tau, R\hat{r}) \qquad (R = \text{constant}) \tag{4.1}$$

$$V^\mu = \gamma(1, \boldsymbol{v}) = \gamma(1, \omega R\hat{\varphi}) \tag{4.2}$$

$$\gamma \equiv (1 - \boldsymbol{v}\cdot\boldsymbol{v})^{-1/2} \tag{4.3}$$

$$\omega \equiv \mathrm{d}\varphi/\mathrm{d}t \tag{4.4}$$

where $(\hat{r}, \hat{\varphi}, \hat{z})$ is a right-handed system of cylindrical coordinates. In this case, the second term in equation (2.4) vanishes. Defining

$$\boldsymbol{Q} = Q^r\hat{r} + Q^\varphi\hat{\varphi} + Q^z\hat{z} \tag{4.5}$$

we then obtain

$$\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}\tau} = \gamma\frac{W(R)}{R}(\boldsymbol{r}\times\boldsymbol{v})\times\boldsymbol{Q} = R\gamma W(R)\omega\hat{z}\times\boldsymbol{Q}. \tag{4.6}$$

Equation (4.6) becomes in component form

$$\frac{dQ^{\varphi}}{d\tau} = R\gamma W(R)\omega Q^{r} \tag{4.7}$$

$$\frac{dQ^{r}}{d\tau} = -R\gamma W(R)\omega Q^{\varphi} \tag{4.8}$$

$$\frac{dQ^{z}}{d\tau} = 0. \tag{4.9}$$

The relevant component Q^{φ} has the solution

$$Q^{\varphi} = Q_0 \sin[RW(R)\varphi] \qquad (Q_0 = \text{constant}) \tag{4.10}$$

where an arbitrary phase has been chosen equal to zero. Substituting equation (4.10) into the scalar isocharge evolution equation (2.4) and rewriting in terms of the azimuthal angle φ gives

$$\frac{dQ}{d\varphi} = Rf(R)Q_0 \sin[RW(R)\varphi] \tag{4.11}$$

$$Q = Q_1 - Q_0 \frac{f(R)}{W(R)} \cos[RW(R)\varphi] \qquad (Q_1 = \text{constant}) \tag{4.12}$$

where $f(R)$ was defined in equation (3.12).

The asymptotic forms of the functions f/W and RW appearing in equations (4.11) and (4.12) are

$$\frac{f(R)}{W(R)} \sim A_0 R \exp(-R) \tag{4.13}$$

$$RW \sim 1 - A_0 R \exp(-R). \tag{4.14}$$

The amplitude of the oscillating term falls off like $R\exp(-R)$ for $R \to \infty$. However, if $R = O(1)$, i.e. near the 'core' of the monopole, the scalar isocharge undergoes oscillations of relative order unity. Similarly, since for $R \gg 1$ the frequency RW is nearly unity, the phase of the oscillations can be thought of, in that case, as undergoing a small secular shift proportional to $R\exp(-R)$ per revolution. For $R = 0(1)$, this phase shift is of the order of unity.

5. Conclusions

We have introduced a scalar, gauge-invariant quantity $Q(\tau)$ (given by equation (2.2)) associated with a point-like isocharge in classical Yang-Mills-Higgs theory, and we have examined the evolution of $Q(\tau)$ in the presence of an 'tHooft-Polyakov monopole. If the point charge creates a weak disturbance ($|q| \ll 1/g$), then we arrive at the following results, to lowest order in the small parameter ε associated with the weakness of the disturbance.

(i) If the distance of closest approach b satisfies $1/(bgF) \ll 1$, then $dQ/d\tau$ vanishes to leading order in $1/(bgF)$, the precise result being given by equations (2.5) and (2.6), respectively, for the cases $\lambda \neq 0$, $\lambda = 0$, where τ is the proper time and λ is the magnitude of the Higgs potential appearing in the Lagrangian (1.2).

(ii) For forced straight-line motion (not necessarily in the far field of the monopole), we have calculated $Q(\tau)$ and find that $\Delta Q \equiv Q(\infty) - Q(-\infty)$ vanishes.

(iii) For the case of forced uniform circular motion with radius R, Q evolves sinusoidally, exhibiting a secular advanced in phase (analogous to the precession of the perihelion of an orbiting body in general relativity) that depends strongly on R. The amplitude and secular advance of Q are significant only if $R = O(1/(fg))$, i.e. in the core region of the monopole.

It is interesting to note that uniform circular motion is also a reasonable approximation to the *unforced* motion of an isocharge in the field of a Julia-Zee dyon [18] for the case in which the charge parameter γ_0 satisfies $\gamma_0 \gg 1$. The above procedure could be carried out with little modification.

Our results complement and augment those of Drechsler [1, 3, 4], Havas [3, 4], Kates [2] and Rosenblum [3, 4] in the sense that the present paper gives a definite answer in a specific problem to the question of whether or not a gauge-invariant measure of isocharge is necessarily constant in classical Yang-Mills-Higgs theories. For our definition of gauge-invariant isocharge, the answer is evidently no. We note that Marciano and Muziwich [19] also obtained results consistent with isocharge changing in the field of a monopole for the case of test Dirac fields.

In a future paper, the time-odd effects on $Q(\tau)$ associated with radiation reaction will be considered.

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